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Methods of Alice physics

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Abstract. Recently there has been a revival of interest in gauge theories with *disconnected* compact gauge groups. Gauge fields such that this disconnectedness has non-trivial physical effects are called Alice configurations. The question of the existence of such configurations is surprisingly intricate, and no complete theory is known. Here we give some simple techniques for establishing the existence of Alice configurations, with an emphasis on practical, readily verifiable conditions requiring a minimum of topological information on the base manifold and the gauge group.

1. Introduction

The idea that gauge groups need not be connected—that they can, in physical parlance, be 'discrete'—dates back to the work of Kiskis [1]. It arose again in connection with the theory of vortices and cosmic strings [2, 3], and then in studies of the role of discrete symmetries in quantum gravity [4, 5]. More recently, the fundamental importance of discrete gauge groups was explicitly recognized in several deep studies of quantum hair on black holes [6, 7]. Yet again, 'statistical hair' [8] is described as a type of 'discrete gauge hair'.

Discrete gauge effects can originate in several different ways, a fact explained with great clarity by Coleman *et al* [6]. Discrete gauge symmetries can make their presence felt very directly if the theory in question is defined on a manifold which is not simply connected, and this is the case we shall consider here. (Our results after section 2 are therefore not relevant, for example, to the gauge configurations considered by Schwartz [2] which are defined on ordinary Minkowski space.) The manifold need not be the four-dimensional spacetime manifold: string and brane theories provide numerous opportunities for non-simply-connected internal spaces to play a role. Parallel transport around non-contractible loops can lead to quite extraordinary physical effects. For example, in an 'Alice universe' [7] a journey around such a loop can convert matter to anti-matter. A question now arises, however: given a specific non-simply-connected manifold M, how can a non-trivial discrete gauge configuration actually be constructed? For example, the Alice universe involves the group Pin(2). Suppose that the topology of spacetime is $\mathbb{R} \times \mathbb{R}P^3$, where $\mathbb{R}P^3$ is the real projective 3-space: how do we determine whether a non-trivial Pin(2) gauge configuration exists on this spacetime? (In fact, as we shall see, *it does not*.) The objective of this work is to give some techniques for constructing non-trivial discrete gauge configurations over specific non-simply-connected manifolds and also to discuss some examples where this *cannot* be done. This will also involve a study of the structure of *disconnected* compact Lie groups.

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Given a compact connected Lie group G_0, we can easily find examples of disconnected groups with G_0 as the identity component: indeed, it is all too easy to find them. For if F is any finite group, then $F \times G_0$ is such a group. This complicates the problem of giving a meaningful classification; however, in section 2 we give a very simple approach which is nevertheless adequate for our purposes. Given G_0, we assign a compact disconnected group with G_0 as identity component to one of three classes, according to the relationship between the 'discrete' and the 'continuous' parts of the group.

However, it is clear that merely extending a group in this way hardly permits us to claim that the corresponding discrete symmetries have been 'gauged'. For that, we need to establish the existence of gauge configurations such that parallel transport actually involves elements of the group outside the identity component. In geometric language, the question is this: given a connected manifold M and a (not necessarily connected) compact Lie group G, can we find a principal bundle over M with a connection having *holonomy group* isomorphic to G? If the bundle is required to be the bundle of orthonormal frames over a Riemannian manifold, and if the torsion form is required to vanish, then this question leads to the celebrated Berger classification [9]. For compact, *simply-connected* M, our question has finally been completely answered in this special case by the work of Joyce [10, 11], though the non-simply-connected case has only been settled for compact manifolds of positive Ricci curvature [12]. Here, however, we shall allow the bundle to be an *arbitrary* gauge bundle. In this case, if G is *connected*, then our question has a very simple answer: subject to mild conditions on M, *any* connected G occurs as the holonomy group of a connection on some bundle over M. (This is a consequence of the Hano–Ozeki–Nomizu theorem [13], which we shall use extensively.) However, if G is *disconnected*, it often cannot be represented as a holonomy group over a given M even if we allow arbitrary bundles. The very *existence* of non-trivial discrete gauge configurations is thus problematic.

It is intuitively clear that a disconnected group cannot occur as a holonomy group over a simply-connected manifold. There must be a match between the size of the *fundamental group* of M and the number of connected pieces comprising G. This observation can be made precise and formulated as a very simple condition on M which must be satisfied if G is to occur as a holonomy group over M. Physical intuition suggests that this matching condition, which we call the *holonomy covering condition*, should be sufficient as well as necessary. This is indeed true in many (but not all) cases; a large class of such cases is described in section 3.

The elementary approach of section 3 fails for some groups. Of these, some can be treated by constructing a special type of function which we call an *antipodal function*. This technique is explained in section 4. Another approach is to impose on M a covering condition stronger than the holonomy covering condition; this is described in section 5. Still another approach is given in section 6. Nevertheless there remain some disconnected groups G and base manifolds M such that all of these techniques fail to allow us to construct a bundle over M having a connection with holonomy group isomorphic to G, even if M satisfies the holonomy covering condition. This leads to the suspicion that, for some M satisfying the holonomy covering condition, there *does not* exist any principal bundle with a connection having holonomy group isomorphic to G. This suspicion is confirmed in section 7; we present many examples of just this kind, where the holonomy covering condition definitely fails to be sufficient. This leads to some surprising conclusions: for example, while Pin(3) occurs as a holonomy group over the real projective space $\mathbb{R}P^3$, its subgroup Pin(2) does not. Again, Pin(2) can be a holonomy group on a bundle over the group manifold $SO(6)/\mathbb{Z}_2$, but *not* on any bundle over $SO(6)$ itself, though $SO(6)$ satisfies the holonomy covering condition for Pin(2).

Some readers will be aware that some of the issues being discussed here can be formulated in a much more sophisticated way, in terms of obstruction theory. Unfortunately, that approach requires a great deal of information on the topology of the base manifold M and the gauge group G (involving certain cohomology groups of M with coefficients twisted by homotopy groups of G). In practice, no single technique is adequate to handle all M and all G. Our emphasis here is on relatively elementary techniques which the reader can easily adapt to high-dimensional base manifolds and gauge groups with complicated topologies (such as the standard group), for which the relevant obstruction classes can be very difficult to obtain. For example, our methods can easily handle the specific case of an $SO(10)$ grand unified theory, breaking to various physically interesting subgroups, defined on *any* homogeneous, locally isotropic cosmological model in *arbitrarily high* dimensions. Again, Anandan [14] has recently discussed quantum mechanics in Alice cosmologies, and our methods immediately establish the existence of the corresponding gauge configurations.

We begin by describing a simple structure theory for disconnected compact Lie groups.

2. Disconnected compact Lie groups

A general theory of disconnected compact Lie groups, concentrating on the semi-simple case, has been given by de Siebenthal [15]. However, the following considerations suffice for our purposes. For any Lie group G, the connected component containing the identity, G_0, is a normal subgroup, and G can be expressed as a disjoint union of connected components

$$G = G_0 \cup c_1 \cdot G_0 \cup c_2 \cdot G_0 \cup \cdots$$

where c_i are not elements of G_0. The number of components is finite if, as we assume henceforth, G is compact. Note that, for each i, $\mathrm{Ad}(c_i)$, which is a conjugation by c_i, is an automorphism of G_0.

Now let G_1 be the subgroup of G consisting of all g such that $\mathrm{Ad}(g)$ is an inner automorphism of G_0. Let g be any element of G_1, and let k be any element of G; then since the inner automorphisms are normal in the group of all automorphisms of G_0, $\mathrm{Ad}(k)\mathrm{Ad}(g)\mathrm{Ad}(k^{-1})$ is inner, and thus kgk^{-1} is an element of G_1. Hence the latter is normal in G, and we have a natural coset decomposition

$$G = G_1 \cup k_1 \cdot G_1 \cup k_2 \cdot G_1 \cup \cdots$$

where each $\mathrm{Ad}(k_i)$ is outer on G_0. Note that if $\mathrm{Ad}(k_i) = \mathrm{Ad}(k_j)$ modulo an inner automorphism, then $k_i \cdot G_1 = k_j \cdot G_1$, so there can be no more terms in this decomposition than there are elements in the outer automorphism group of G_0.

Given any group G, and subgroups A and B, the set of all products $A \cdot B$, is a subgroup if $A \cdot B = B \cdot A$. Let K be the group generated by k_i above: then we have $G = K \cdot G_1$. Note that K and G_1 may have a non-trivial intersection, and also that the structure of K depends on the choice of k_i. This last observation means that, in writing $G = K \cdot G_1$, we are giving a specific *presentation* of G. We shall resolve this ambiguity here by giving a *specific, fixed choice* of k_i for each G_0; but the reader should bear in mind that the presentations we give are not unique. This is not a matter of concern in practice.

For G_1, we have

$$G_1 = G_0 \cup h_1 \cdot G_0 \cup h_2 \cdot G_0 \cup \cdots$$

where each $\mathrm{Ad}(h_i)$ is inner on G_0; hence h_i can be chosen so as to commute with every element of G_0, and we shall always choose them in this way. If H is the group generated by some specific set of h_i, then we have $G_1 = H \cdot G_0$, where again H may intersect G_0 non-trivially (in its centre).

To summarize then, all of the complications in the structure of a compact disconnected Lie group G arise either from the existence of outer automorphisms of G_0, or from non-trivial intersections of K or H with the centre of G_0. Thus if G_0 is the exceptional group E_8, which has no outer automorphisms and a trivial centre, then the only compact disconnected groups with E_8 as their identity component are direct products of finite groups with E_8. For E_7 the situation is slightly more complex, since E_7 has a centre isomorphic to $\mathbb{Z}_2$, while E_6 is still more complicated, since its centre is $\mathbb{Z}_3$ and it has outer automorphisms.

These considerations lead to the following classification. It is a classification of *presentations* of disconnected compact Lie groups: that is, we suppose that generators of K and H are specified and fixed. (The classification is thus *not* a precise classification of G according to its abstract isomorphism class. For our purposes, the classification of presentations is more useful.) There are three types.

Type I: K is trivial. Here $G = G_1 = H \cdot G_0$, with each element of the finite group H commuting with every element of G_0. If G_0 has no outer automorphism, then every G is of this kind: this is the case for $\mathrm{Spin}(n)$ and $SO(n)$ when n is odd, for all of the symplectic groups $\mathrm{Sp}(n)$, and for all of the exceptional groups other than E_6. Note that every finite group is of this type (take G_0 trivial) and so is every compact connected group (take H trivial). As we have emphasized the only complication here is the possibility of a non-trivial intersection of H with the centre of G_0. For example, let $G_0 = SU(2)$ and let $H = \mathbb{Z}_4$, generated by z such that z commutes with every element of $SU(2)$, and $z^2 = -I_2$, where I_n is the $(n \times n)$ identity matrix. Then $\mathbb{Z}_4$ and $SU(2)$ intersect in $\{\pm I_2\}$, and $\mathbb{Z}_4 \cdot SU(2)$ is a group with two (not four) connected components. This group is *not* isomorphic to $\mathbb{Z}_2 \times SU(2)$, which shows the need for caution even in these simple cases. (This group occurs as the linear holonomy group of any Enriques surface [9] endowed with a Yau metric.)

Type II: $K \neq \{e\}$, $K \cap G_0 = \{e\}$. Here e is the identity element of G. In this case K is not trivial, but there are no complications due to non-trivial intersections with G_0. The most important examples here arise from the unitary groups $U(n)$, the Spin groups $\mathrm{Spin}(2n)$, the special orthogonal groups $SO(2n)$, and the exceptional group E_6. Consider, for example, the orthogonal groups $O(2n)$, which we present as follows. For each integer n, let A_{2n} be the $((2n) \times (2n))$ diagonal matrix

$$A_{2n} = \mathrm{diag}(-1, 1, 1, 1 \ldots) \qquad n \quad \text{even}$$
$$A_{2n} = \mathrm{diag}(1, -1, 1, -1, \ldots) \qquad n \quad \text{odd}$$

so that $\det A_{2n} = -1$ for all n. Then $\mathrm{Ad}(A_{2n})$ is an outer automorphism of $SO(2n)$, and $O(2n) = SO(2n) \cup A_{2n} \cdot SO(2n)$; clearly K is just $\{I_{2n}, A_{2n}\}$, and I_{2n} is the only common element of K and $SO(2n)$. Thus $O(2n)$ is of type II. Similarly, $U(n)$ is the identity component of a type II group, with two components, which is a semi-direct product of $U(n)$ with $\mathbb{Z}_2$ (the relevant outer automorphism in this case being complex conjugation). Another example of this type is the following: the grand unification group [16] $\mathrm{Spin}(10)$ contains the colour $SU(3)$ and an element γ of order 4 such that $\mathrm{Ad}(\gamma)$ induces complex conjugation on $SU(3)$, but γ^2 is not an element of $SU(3)$. Here $K = \mathbb{Z}_4$, and H is the $\mathbb{Z}_2$ subgroup of $\mathbb{Z}_4$; K intersects $SU(3)$ trivially, and G is a group, with four connected components, of type II. (This is the gauge group of 'Alice chromodynamics' in the context of $\mathrm{Spin}(10)$ grand unification.)

Type III: $K \neq \{e\}$, $K \cap G_0 \neq \{e\}$. The groups of this type involve both kinds of complication, and they are the ones which present the most serious difficulty in holonomy theory. The key example here, both physically [7] and geometrically, is the group Pin(2), the non-trivial double cover of the orthogonal group $O(2)$. Let e_1 and e_2 generate the Clifford algebra on $\mathbb{R}^2$; then $\mathrm{Ad}(e_1)$ is an outer automorphism of Spin(2), and $(e_1)^2 = -1$, so that if we present Pin(2) as

$$\mathrm{Pin}(2) = \mathrm{Spin}(2) \cup e_1 \cdot \mathrm{Spin}(2)$$

then $K = \mathbb{Z}_4$ and $K \cap G_0 = \{\pm 1\}$. Thus Pin(2) is of type III.

There are three ways to generalize this example. The first is to consider

$$\mathrm{Pin}(2n) = \mathrm{Spin}(2n) \cup e_1 \cdot \mathrm{Spin}(2n)$$

for all $n \geqslant 1$. The second way is to recall that Spin(2) is isomorphic to the unitary group $U(1)$. Let α_n be such that $\mathrm{Ad}(\alpha_n)$ induces complex conjugation on $U(n)$, and $\alpha_n^2 = -I_n$; then

$$\mathbb{Z}_4 \cdot U(n) = U(n) \cup \alpha_n \cdot U(n)$$

is a group of type III with two connected components. Clearly $\mathbb{Z}_4 \cdot U(1) = \mathrm{Pin}(2)$. Third, we can regard $O(2n)$, in the presentation given earlier, as the real subgroup of $U(2n)$ and note that the matrices $\mathrm{i}A_{2n}$ all belong to $U(2n)$. Then the groups

$$\mathbb{Z}_4 \cdot SO(2n) = SO(2n) \cup \mathrm{i}A_{2n} \cdot SO(2n)$$

are two-component subgroups of $U(2n)$ of type III. Again, recalling that $\mathrm{Spin}(2) = SO(2)$, we verify that $\mathbb{Z}_4 \cdot SO(2) = \mathbb{Z}_4 \cdot U(1) = \mathrm{Pin}(2)$. Summarizing, we have in $\mathrm{Pin}(2n)$, $\mathbb{Z}_4 \cdot U(n)$ and $\mathbb{Z}_4 \cdot SO(2n)$, three infinite families of groups of type III, all three having $\mathrm{Pin}(2) = \mathbb{Z}_4 \cdot U(1) = \mathbb{Z}_4 \cdot SO(2)$ as their only common member.

When considering the holonomy theory of type III groups, we shall concentrate our attention on these three families, for the following reasons. Recall that any compact disconnected group has the form $G = K \cdot G_1$, $G_1 = H \cdot G_0$. If G is a holonomy group of a connection in some gauge bundle, then K will have a clear physical significance: since, for each k_i, $\mathrm{Ad}(k_i)$ has a non-trivial effect on maximal tori in G_0, parallel transport involving K will affect some of the charges of the theory, this being the distinctive property of Alice gauge configurations. However, parallel transport involving H has no such effect and so the physical significance of H is not clear. If H is trivial, then G is called [15] a *natural extension* of G_0. Now de Siebenthal shows [15] that the only natural extensions of $SO(2n)$ are $O(2n)$ and $\mathbb{Z}_4 \cdot SO(2n)$ and that the only natural extensions of $\mathrm{Spin}(2n)$ (except when $n = 4$, where triality leads to further groups, all of type II) are $\mathrm{Pin}(2n)$ and a type II group of the form $\mathbb{Z}_2 \cdot \mathrm{Spin}(2n)$; and similarly one can prove that $\mathbb{Z}_2 \cdot U(n)$ and $\mathbb{Z}_4 \cdot U(n)$ are the only natural extensions of $U(n)$. Thus, there are good physical and mathematical reasons for concentrating on $\mathrm{Pin}(2n)$, $\mathbb{Z}_4 \cdot U(n)$ and $\mathbb{Z}_4 \cdot SO(2n)$. In fact, of these, $\mathrm{Pin}(2n)$ and $\mathbb{Z}_4 \cdot U(n)$ are of most interest in physics, since they (and similar groups) occur naturally as subgroups of spin groups. For example, Pin(2) arises in the work of Preskill and Krauss [7] as a subgroup of Spin(3), and the electromagnetic subgroup of the Spin(10) grand unification group may be regarded as the identity component of a certain Pin(2) subgroup of Spin(10). (When $SO(n)$ is mentioned in the physics literature, $\mathrm{Spin}(n)$ is usually intended.) In short, one should think of Pin(2) as the natural 'disconnected version' of $U(1)$ and of $\mathrm{Pin}(2n)$ and $\mathbb{Z}_4 \cdot U(n)$ as generalizations of Pin(2).

3. The holonomy covering condition and types I and II

Let P be a principal fibre bundle with structural group G over a connected base manifold M, and suppose that there is a connection on P with the holonomy group isomorphic to G.

There is a natural homomorphism [13] from the fundamental group $\pi_1(M)$ onto G/G_0, so $\pi_1(M)$ has a normal subgroup N such that $\pi_1(M)/N = G/G_0$. Thus M has a non-trivial connected covering $\overline{M}$ such that $\overline{M}/(G/G_0) = M$. In general, if a manifold M has a connected covering $\overline{M}$ such that there exists a discrete group D acting freely and properly discontinuously on $\overline{M}$, with $\overline{M}/D = M$, then we shall say that M has a D-covering. The above remarks yield the following simple but crucial result.

Theorem 1. For a Lie group G to occur as a holonomy group of a connection on a principal bundle over M, it is necessary for M to have a (G/G_0)-covering.

For example, we have $\mathrm{Pin}(2)/\mathrm{Spin}(2) = \mathbb{Z}_2$, and so M must have a *double cover* ($\overline{M}/\mathbb{Z}_2 = M$) if there is to be any chance of constructing over M, a bundle with a connection having Pin(2) holonomy. The obvious way for this condition to fail is for $\pi_1(M)$ to be of odd order, but that is not the only way: for example, the Poincaré homology sphere [17] has a fundamental group of order 120, but it has no double cover. For another example, consider $Q_8 \cdot SU(2)$, where Q_8 is the quaternionic group of order 8, and $Q_8 \cap SU(2) = \{\pm I_2\}$. Here $(Q_8 \cdot SU(2))/SU(2) = \mathbb{Z}_2 \times \mathbb{Z}_2$ and in fact $Q_8 \cdot SU(2)$ is the linear holonomy group of Hitchin's [18] manifold $K3/[\mathbb{Z}_2 \times \mathbb{Z}_2]$.

We refer to the condition that M should have a (G/G_0)-covering as the *holonomy covering condition* for the pair (M, G). The key question now, of course, is that of whether this condition is *sufficient* as well as necessary. In general, the answer is 'no'; but for a surprising number of groups, it is 'yes'. We have the following result.

Theorem 2. Let M be a connected, paracompact manifold of dimension at least 2, and let G be a compact Lie group having a presentation of type I with H cyclic, or of type I with H intersecting G_0 trivially, or of type II. There exists a principal fibre bundle over M with a connection having holonomy group isomorphic to G if and only if M satisfies the holonomy covering condition with respect to G.

The proof is somewhat similar to that of theorem 3, and so we shall omit it. Note that the restriction on the dimension is understandable if we recall that the curvature forms, being two-forms, must vanish if M is one-dimensional, and so only finite groups are possible (in the compact case) if M is one-dimensional.

This result settles our question for the most interesting groups of type I and for all groups of type II. Notice that since every compact connected group is of type I, theorem 2 is a generalization of the statement [13] that every such group occurs as a holonomy group over any paracompact connected manifold with dimension at least two. Similarly, every finite group is of type I and can be handled by this theorem. Again, the orthogonal group $O(2n)$ (type II) occurs as a holonomy group in some bundle over *any* M which has a double cover, whether it be orientable or not, provided that it is paracompact and $\dim(M) \geqslant 2$.

In view of theorem 2, we concentrate henceforth on disconnected groups of type III. We begin with a technique which allows us to deal with certain pairs (M, G) when G is of type III.

4. Antipodal functions and pin-like groups

As we explained in section 2, we shall concentrate on the groups $\mathrm{Pin}(2n)$, $\mathbb{Z}_4 \cdot U(n)$ and $\mathbb{Z}_4 \cdot SO(2n)$, for all $n \geqslant 1$. All of these are *Pin-like groups*, in the following sense: all have the form

$$G = G_0 \cup \gamma \cdot G_0$$

where $\mathrm{Ad}(\gamma)$ induces some specific outer automorphism of order 2 on G_0, which fixes a distinguished element -1 in the centre of G_0, and where $\gamma^2 = -1$. We denote the outer automorphism by $g \to \overline{g}$; recall that this is conjugation by e_1 for $\mathrm{Spin}(2n)$, complex conjugation for $U(n)$ and conjugation by A_{2n} for $SO(2n)$. Note that when n is even, $\mathbb{Z}_4 \cdot SU(n)$ is well defined and Pin-like.

For Pin-like groups, we have the following construction. For all of these groups, the holonomy covering condition is just the requirement that M should have a double cover, $\overline{M}$. Thus, we assume that $\overline{M}$ exists; let μ be the fixed-point-free involution on $\overline{M}$ generating $\mathbb{Z}_2$ such that $\overline{M}/\mathbb{Z}_2 = M$. Let G be any Pin-like group. A smooth map $f : \overline{M} \to G_0$ will be called an *antipodal function* on $\overline{M}$ with respect to G_0 if for all $x \in \overline{M}$

$$f(\mu(x)) = -(\overline{f(x)})^{-1}.$$

To understand the reason for the terminology, we take M to be the real projective space $\mathbb{R}P^n$ and let G be $\mathrm{Pin}(2)$. Then $\overline{M}$ is the sphere S^n, μ is the antipodal map and G_0 is $\mathrm{Spin}(2)$, in which every element satisfies $(\overline{g})^{-1} = g$. Furthermore, $\mathrm{Spin}(2)$ is homeomorphic to the one-sphere and the map $a : g \to -g$ is antipodal on S^1; hence an antipodal map f in this case is one which makes the following diagram commute

$$\begin{array}{ccc} S^n & \xrightarrow{f} & S^1 \\ \mu \downarrow & & \downarrow a \\ S^n & \xrightarrow{f} & S^1 \end{array}$$

This is the usual definition of an antipodal function from one sphere to another [17].

If they exist, antipodal functions are very useful, as the following theorem shows.

Theorem 3. Let M be a connected, paracompact manifold of dimension at least 2, and let G be a Pin-like group. Suppose that M has a non-trivial double cover $\overline{M}$, and that there exists an antipodal function on $\overline{M}$ with respect to G_0. Then, there exists a principal fibre bundle over M with a connection having holonomy group isomorphic to G.

Proof. Set $P = \overline{M} \times G_0$, and define an action of G on P as follows. Recall that every element of G is either of the form $g \in G_0$ or of the form $\gamma g \in \gamma \cdot G_0$. For any $(x, s) \in P$, let R_g and $R_{\gamma g}$ be defined by

$$R_g : (x, s) \to (x, sg)$$

$$R_{\gamma g} : (x, s) \to (\mu(x), f(x)\overline{s}g)$$

where μ is the canonical involution on $\overline{M}$ and f is the given antipodal function on $\overline{M}$. We claim that this is a right action by G: that is, if g_1 and g_2 are any elements of G_0, then $R_{g_1}R_{g_2} = R_{g_2 g_1}$, $R_{\gamma g_1}R_{g_2} = R_{g_2\gamma g_1}$, $R_{g_1}R_{\gamma g_2} = R_{\gamma g_2 g_1}$ and $R_{\gamma g_1}R_{\gamma g_2} = R_{\gamma g_2 \gamma g_1}$. The first and third are clear. For the second

$$R_{\gamma g_1}R_{g_2}(x, s) = R_{\gamma g_1}(x, sg_2) = (\mu(x), f(x)\overline{s}\overline{g}_2 g_1)$$

which is the effect of $\gamma\overline{g}_2 g_1$; this is correct, since $\gamma\overline{g}_2 g_1 = g_2\gamma g_1$. Again, we have

$$R_{\gamma g_1}R_{\gamma g_2}(x, s) = R_{\gamma g_1}(\mu(x), f(x)\overline{s}g_2) = (x, f(\mu(x))\overline{f(x)}s\overline{g}_2 g_1)$$

which, by definition of f, is $(x, -s\overline{g}_2 g_1)$; this is correct, since $\gamma g_2 \gamma g_1 = \gamma^2 \overline{g}_2 g_1 = -\overline{g}_2 g_1$. Thus, the action of G is indeed to the right and, since μ has no fixed point, this action is free. Regarding $\overline{M}$ as P/G_0, we see that $P/G = M$. Let V be an open set in M, and σ a local section of $\overline{M}$ as a locally trivial bundle over M. If $\pi : P \to M$ is the projection, $\pi^{-1}(V)$ is the disjoint union of $\sigma(V) \times G_0$ with $\mu\sigma(V) \times G_0$. That is, every point in $\pi^{-1}(V)$ is either of the form $(\sigma(x), g)$ or of the form $(\mu\sigma(x), g)$, where $x \in V$ and $g \in G_0$. Define a map $\phi : \pi^{-1}(V) \to G$ by

$$\phi : (\sigma(x), g) \to \overline{f(\sigma(x))}g$$
$$\phi : (\mu\sigma(x), g) \to \gamma g.$$

We leave it to the reader to confirm that this defines an isomorphism of $\pi^{-1}(V)$ with $V \times G$, so that P is locally trivial over M. Thus P is a principal G bundle over M. Now the Hano–Ozeki–Nomizu theorem [13] states that if P is a connected G bundle over a paracompact base M with $\dim(M) \geqslant 2$, then there exists a connection on P with holonomy group isomorphic to G. As P is evidently connected in our case, the proof is now complete. □

A careful examination of the proof reveals the reasons for the fact that types I and II are more tractable than type III. Consider, for example, the type I group $\mathbb{Z}_4 \cdot SU(2) = SU(2) \cup z \cdot SU(2)$. Here we can set $P = \overline{M} \times SU(2)$ and define the action of z by

$$R_z : (x, s) \to (\mu(x), ts)$$

where t is an element of $SU(2)$ such that $t^2 = -I_2$; following the proof of theorem 3, one now can prove theorem 2 in this case. Consider, on the other hand, the type II group $O(2) = SO(2) \cup A \cdot SO(2)$, where $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Here we set $P = \overline{M} \times SO(2)$ and define the action of A by

$$R_A : (x, s) \to (\mu(x), \overline{s})$$

where $\overline{s} = \mathrm{Ad}(A)s$. One can now prove theorem 2 for type II. However, if we attempt to combine these ideas to deal with the type III group $\mathrm{Pin}(2) = \mathrm{Spin}(2) \cup e_1 \cdot \mathrm{Spin}(2)$, we find that $(x, s) \to (\mu(x), t\overline{s})$ (for some $t \in \mathrm{Spin}(2)$ satisfying $t^2 = -1$) does *not* define an action by e_1 : for $(x, s) \to (\mu(x), t\overline{s}) \to (x, t\overline{t}s) = (x, s)$ is the identity map, but $(e_1)^2 \neq 1$. Antipodal functions are designed to deal with this problem.

Let us consider an example where theorem 3 can be used. Let a_1, a_2, a_3 be linearly independent vectors in $\mathbb{R}^3$, with a_1 orthogonal to a_2 and a_3 and let t_1, t_2, t_3 be the corresponding translations. Let α be the affine map on $\mathbb{R}^3$ defined by $a_1 \to a_1, a_2 \to -a_2, a_3 \to -a_3$, followed by translation through $a_1/2$. Then the group Γ generated by t_1, t_2, t_3 and α acts freely and properly discontinuously on $\mathbb{R}^3$, and $\mathbb{R}^3/\Gamma$ is a compact flat manifold [19]. This manifold is a three-dimensional version of the Klein bottle; it can also be regarded as $T^3/\mathbb{Z}_2$, where T^3 is the 3-torus. We claim that *every group of the form* $\mathbb{Z}_4 \cdot U(n)$ *and every group of the form* $\mathbb{Z}_4 \cdot SO(2n)$, *occurs as a holonomy group* on some bundle over $\mathbb{R}^3/\Gamma$. To prove this we must exhibit the corresponding antipodal functions on T^3, the double cover of $\mathbb{R}^3/\Gamma$.

Let x, y, z be the usual coordinates on $\mathbb{R}^3$, and take a_1, a_2, a_3 to be the corresponding orthonormal basis. For each of the groups $U(n)$, $SO(2n)$ (n even) and $SO(2n)$ (n odd) we define a function $\hat{f}$ on $\mathbb{R}^3$, taking its values in those groups, as follows

$$\begin{aligned}
\hat{f}(x, y, z) &= \exp[2\pi \mathrm{i} x] \cdot I_n \qquad \text{for } U(n) \\
&= \mathrm{diag}\left[\begin{pmatrix} \cos 2\pi x & -\sin 2\pi x \\ \sin 2\pi x & \cos 2\pi x \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ldots \right] \\
&\quad \text{for } SO(2n) \quad n \text{ even} \\
&= \mathrm{diag}\left[\begin{pmatrix} \cos 2\pi x & -\sin 2\pi x \\ \sin 2\pi x & \cos 2\pi x \end{pmatrix}, \begin{pmatrix} \cos 2\pi x & -\sin 2\pi x \\ \sin 2\pi x & \cos 2\pi x \end{pmatrix}, \ldots \right] \\
&\quad \text{for } SO(2n) \quad n \text{ odd.}
\end{aligned}$$

Now clearly $t_i^* \hat{f} = \hat{f}$ for all i and each $\hat{f}$, so we obtain three projected functions f on T^3. Let μ be the involution on T^3 induced by α. We leave it to the reader to verify, using the specific forms we have given earlier for the outer automorphisms (complex conjugation for $U(n)$, conjugation by A_{2n} for $SO(2n)$) that all three functions f satisfy the defining relation for antipodal functions. Then theorem 3 gives us $\mathbb{Z}_4 \cdot U(n)$ and $\mathbb{Z}_4 \cdot SO(2n)$ as holonomy groups on bundles over $\mathbb{R}^3/\Gamma$. The quantum mechanics of gauge fields on manifolds of this kind has recently been discussed, *assuming* the existence of Alice gauge configurations, in an interesting paper by Anandan [14].

For certain groups, an even simpler way of constructing antipodal functions is possible. Suppose that $f : \overline{M} \to G_0$ is the *constant* function, $f(x) = \phi$ for all x. Then f is antipodal if G_0 contains an element ϕ such that $\phi = -(\overline{\phi})^{-1}$. Such an element may be called an *antipodal constant*. The problem here is that not every G_0 has an antipodal constant. We have the following result.

Theorem 4. The group Spin$(2n)$ has antipodal constants if and only if $n \geqslant 2$. The group $U(n)$ has antipodal constants if and only if n is even. The group $SO(2n)$ has no antipodal constants, for any n.

Proof. Let Spin(n) be defined, as usual, in terms of the Clifford algebra generated by $\{e_i\}$, $i = 1 \ldots n$. If $n \geqslant 3$, Spin(n) contains an element e_2e_3; setting $\phi = e_2e_3$, we have

$$\phi\overline{\phi} = -e_2e_3e_1e_2e_3e_1 = -1$$

so ϕ is antipodal. By contrast, in Spin(2), every element satisfies $g\overline{g} = 1$, so there is no antipodal constant. For $U(n)$, if n is even we can set

$$\phi = \mathrm{diag}\left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ldots\right]$$

and this is antipodal; this also works for $SU(n)$, n even. However, if n is odd then an antipodal constant ϕ in $U(n)$ would satisfy $\phi\overline{\phi} = -I_n$ and so $\delta = \det(\phi)$ would satisfy $\delta\overline{\delta} = -1$, which is impossible. For $SO(2n)$, we use de Siebenthal's theorem [15] on conjugacy in compact disconnected Lie groups: this states that if $c_i \cdot G_0$ is any connected component of such a group, then each element of $c_i \cdot G_0$ can be expressed as gc_itg^{-1} for some $g \in G_0$, where t is an element of a maximal torus of the centralizer of c_i in G_0. Let ϕ be an antipodal constant in $SO(2n)$; then $\phi A_{2n}\phi A_{2n} = -I_{2n}$. Now the matrix $\psi = \phi A_{2n}$ satisfies $\psi^2 = -I_{2n}$, and ψ is an element of the connected component $A_{2n} \cdot SO(2n)$ in $O(2n)$. Thus, for some B in $SO(2n)$

$$\psi = BA_{2n}DB^{-1}$$

where D belongs to a maximal torus of the centralizer of A_{2n} in $SO(2n)$. If n is even, this centralizer consists of matrices of the form diag$[\pm 1, E]$, where E belongs to $O(2n-1)$, and so $D^2 = \mathrm{diag}[1, E^2]$. But since $\psi^2 = -I_{2n}$, and $(A_{2n}D)^2 = A_{2n}^2 D^2 = D^2$, we have $D^2 = -I_{2n}$, a contradiction. A similar contradiction arises if n is odd. This completes the proof. □

Corollary 4.1. Let M be any connected, paracompact manifold of dimension at least two and let G be Pin(n), $n \geqslant 3$, or $\mathbb{Z}_4 \cdot U(n)$, n even, or $\mathbb{Z}_4 \cdot SU(n)$, n even. There exists a principal bundle over M with a connection having a holonomy group isomorphic to G if and only if M has a non-trivial double cover.

Thus, for example, every $\mathrm{Pin}(n), n \geqslant 3$, occurs as a holonomy group over any real projective space $\mathbb{R}P^m$, and every Pin-like group occurs as a holonomy group over the flat manifold $\mathbb{R}^3/\Gamma$ described earlier.

Useful though they are, antipodal functions are not always easily constructed, and in fact in some cases they do not exist. Take $M = \mathbb{R}P^n$ and $G = \mathrm{Pin}(2)$: then, as we explained earlier, an antipodal function in our sense is an antipodal map, in the usual sense [17] from S^n to S^1. However, the Borsuk–Ulam theorem [17] is precisely the statement that antipodal maps from S^n to S^m, $n > m$, do not exist. Hence, the present method fails in this case.

We can summarize as follows. The method of antipodal functions has allowed us to complete the holonomy existence theory of $\mathrm{Pin}(n), n \geqslant 3$, of $\mathbb{Z}_4 \cdot U(n)$, n even and of $\mathbb{Z}_4 \cdot SU(n)$, n even; in all these cases, the holonomy covering condition is sufficient as well as necessary. The method also has the following property: if it can be made to work for $\mathrm{Pin}(2)$, then it can be made to work for $\mathbb{Z}_4 \cdot U(n)$, n odd and for $\mathbb{Z}_4 \cdot SO(2n)$. In this way, we were able to exhibit these groups as holonomy groups over the flat compact manifold $\mathbb{R}^3/\Gamma$. Thus, $\mathrm{Pin}(2)$ is the crucial case. Unfortunately, the method can definitely fail for $\mathrm{Pin}(2)$: the Borsuk–Ulam theorem implies that it must fail for $M = \mathbb{R}P^n$.

5. Stronger covering conditions

Recall [19] that the quaternionic groups are finite groups Q_{4n} of order $4n$, $n > 1$, generated by two elements a and b satisfying

$$a^n = b^2 \qquad bab^{-1} = a^{-1}.$$

These relations imply $b^4 = 1$, so we define $Q_4 = \mathbb{Z}_4$. We shall say that M has a (non-trivial) *quaternionic cover* of order $4n$ if there exists a connected manifold $\overline{M}$ on which Q_{4n} acts freely, with $\overline{M}/Q_{4n} = M$. The existence of a quaternionic cover is a stronger condition than the existence of a double cover, in the sense that the former entails the latter, but not the reverse. We have the following result.

Theorem 5. Let M be a connected, paracompact manifold, of dimension at least two, with a quaternionic cover of order $4n$. Let G be Pin-like, and suppose that G_0 contains an element p_{2n} such that $(p_{2n})^n = -1$ and $\overline{p_{2n}} = (p_{2n})^{-1}$. Then there exists a principal bundle over M with a connection having a holonomy group isomorphic to G.

Proof. Let Q_{4n} act on $\overline{M}$, the quaternionic cover, through maps $a : \overline{M} \to \overline{M}$ and $b : \overline{M} \to \overline{M}$ satisfying the above relations. Now define a $\mathbb{Z}_{2n}$ action on $\overline{M} \times G_0$ by

$$(x, s) \to (a(x), p_{2n}s).$$

This action is evidently free. We denote the elements of $P = [\overline{M} \times G_0]/\mathbb{Z}_{2n}$ by pairs $\{x, s\}$. Now define an action by G on P as follows. If $g \in G_0$, set

$$R_g : \{x, s\} \to \{x, sg\}$$

while if $\gamma g \in \gamma \cdot G_0$, set

$$R_{\gamma g} : \{x, s\} \to \{b(x), \overline{s}g\}.$$

This is well defined, for $\{a(x), p_{2n}s\}$ is mapped by R_g to $\{a(x), p_{2n}sg\} = \{x, sg\}$ and

$$R_{\gamma g}\{a(x), p_{2n}s\} = \{ba(x), \overline{p_{2n}s}g\} = \{a^{-1}b(x), (p_{2n})^{-1}\overline{s}g\}$$

which is $\{b(x), \overline{s}g\}$. Now we have

$$R_{\gamma g_1} R_{g_2}\{x, s\} = \{b(x), \overline{sg}_2 g_1\}$$

as expected if the action of G is to the right, since $g_2\gamma g_1 = \gamma\overline{g}_2 g_1$. Similarly

$$\begin{aligned} R_{\gamma g_1} R_{\gamma g_2}\{x, s\} &= R_{\gamma g_1}\{b(x), \overline{s} g_2\} = \{b^2(x), s\overline{g}_2 g_1\} = \{a^n(x), s\overline{g}_2 g_1\} \\ &= \{a^{2n}(x), (p_{2n})^n s\overline{g}_2 g_1\} = \{x, -s\overline{g}_2 g_1\} \end{aligned}$$

in agreement with $\gamma g_2 \gamma g_1 = \gamma^2 \overline{g}_2 g_1 = -\overline{g}_2 g_1$. Thus G acts to the right. If R_g had a fixed point, so would some non-trivial power of a; similarly, the fact that no other element of Q_{4n} (except the identity) has a fixed point on $\overline{M}$ means that $R_{\gamma g}$ has no fixed point on P. It is straightforward to show that $P/G = M$, and that P is locally trivial. Since $\overline{M} \times G_0$ is connected, so is its quotient, P. The Hano–Ozeki–Nomizu theorem now yields the existence of the desired connection, and this completes the proof. □

Obviously, if $n = 1$, we can take $p_2 = -1$ for every Pin-like group, so we have the following corollary.

Corollary 5.1. Let M be a connected, paracompact manifold, of dimension at least two, with a connected cover $\overline{M}$ such that $\overline{M}/\mathbb{Z}_4 = M$. Then any Pin-like group occurs as a holonomy group of some connection on some principal bundle over M.

Again, if we define

$$p_{2n} = \operatorname{diag}\left[\begin{pmatrix} \cos(\pi/n) & -\sin(\pi/n) \\ \sin(\pi/n) & \cos(\pi/n) \end{pmatrix}, \begin{pmatrix} \cos(\pi/n) & -\sin(\pi/n) \\ \sin(\pi/n) & \cos(\pi/n) \end{pmatrix}, \ldots \right]$$

then p_{2n} is an element of $SO(2m)$, for any $m \geqslant 1$. Let m be odd; then $A_{2m} p_{2n} A_{2m}^{-1} = p_{2n}^{-1}$ and $(p_{2n})^n = -I_{2m}$. Thus, we have the following consequence of theorem 5.

Corollary 5.2. Let M be a connected, paracompact manifold, of dimension at least two, with a quaternionic cover $\overline{M}$ of any order. Then, there exists a principal bundle over M with a connection having a holonomy group isomorphic to $\mathbb{Z}_4 \cdot SO(2m)$, for any odd m. (Note that this includes $\mathbb{Z}_4 \cdot SO(2) = \mathrm{Pin}(2)$.)

Notice that the second corollary cannot be derived from the first, since the existence of a quaternionic cover does not (always) imply the existence of a $\mathbb{Z}_4$ cover.

These results can be used in cases where the method of antipodal functions fails. For example, corollary 5.1 implies that there is a principal bundle over the lens space [19] $S^3/\mathbb{Z}_4$ with a connection having Pin(2) as holonomy group. The double cover here is $\mathbb{R}P^3$, and it is possible to extend the Borsuk–Ulam argument to show that there is *no* antipodal function on $\mathbb{R}P^3$.

An interesting source of examples of manifolds with quaternionic covers is provided by the homogeneous, locally isotropic Riemannian three-folds. If they are not simply connected, such manifolds [19] are either flat and of the form $\mathbb{R} \times T^2$, $\mathbb{R}^2 \times S^1$, T^3, or they are of the form S^3/Γ, where Γ is a finite group of unit quaternions. Thus S^3/Q_{4n} is a homogeneous, locally isotropic Riemannian manifold—and therefore a possible candidate for the spatial sections of a cosmological model—for all n, and Pin(2) occurs as a holonomy group over all of these spaces. The methods of this section, combined with results to be given later, can actually be used to give a complete analysis of Alice cosmologies with physically interesting disconnected gauge groups.

In short, then, we find that stronger covering conditions can be sufficient to ensure the existence of connections with Pin-like holonomy groups. These conditions, however, are not *necessary*, for the manifold $\mathbb{R}^3/\Gamma$ discussed earlier has no quaternionic cover, and yet every Pin-like group occurs as a holonomy group over $\mathbb{R}^3/\Gamma$. On the other hand, as we shall see, the holonomy covering condition (in this case, the existence of a double cover) definitely fails to be sufficient in some cases. Thus, covering conditions alone cannot always characterize the manifolds on which a given disconnected group occurs as a holonomy group.

6. Reduction to a quotient

If G is a disconnected Lie group and N is a connected normal subgroup of G, then G/N is another disconnected group. If one can solve the problem of exhibiting G as a holonomy group over a given manifold, one might hope to use this to solve the analogous problem for G/N, or *vice versa*. In this section we shall explain the relevant techniques through a specific example.

Theorem 6. Let M be a paracompact manifold. There exists a principal fibre bundle over M with a connection having holonomy $\mathbb{Z}_4 \cdot U(n)$, n odd, if and only if there exists a bundle over M with a connection having holonomy $\mathrm{Pin}(2)$.

Proof. Let P be a principal bundle over M with a connection having holonomy group isomorphic to $\mathbb{Z}_4 \cdot U(n)$, n odd. By the holonomy reduction theorem [13], we may assume without loss of generality that the structural group of P is $\mathbb{Z}_4 \cdot U(n)$. Since the determinant of the complex conjugate of a matrix is the complex conjugate of the determinant, $SU(n)$ is normal in $\mathbb{Z}_4 \cdot U(n)$, and $[\mathbb{Z}_4 \cdot U(n)]/SU(n)$ is isomorphic to $\mathrm{Pin}(2)$. (This is where we use the fact that n is odd, for in that case $SU(n)$ does not contain $(-I_n)$ and so $\mathbb{Z}_4$ and $SU(n)$ intersect trivially; thus, when we take the quotient, $\mathbb{Z}_4$ is not affected. If n is even, then in fact $[\mathbb{Z}_4 \cdot U(n)]/SU(n) = O(2)$, not $\mathrm{Pin}(2)$. Thus, corollary 4.1 is not useful here.) It is not difficult to show that $Q = P/SU(n)$ is a principal $\mathrm{Pin}(2)$ bundle over M, and that the projection $P \to Q$ is a bundle homomorphism. The given connection on P pushes forward to a connection on Q, and the relevant connection mapping theorem [13] states that the group homomorphism $\mathbb{Z}_4 \cdot U(n) \to \mathrm{Pin}(2)$ maps the holonomy group of the connection on P onto the holonomy group of the connection on Q. The latter must therefore be isomorphic to $\mathrm{Pin}(2)$. Conversely, let Q be a bundle over M with a connection having holonomy $\mathrm{Pin}(2)$; again, we may take it that $\mathrm{Pin}(2)$ is the structural group of Q. Set $\hat{P} = Q \times SU(n)$, n odd. Now $\mathrm{Spin}(2)$ contains the cyclic group $\mathbb{Z}_n$, and since Q is a $\mathrm{Pin}(2)$ bundle, we have a fixed-point-free action to the right by $\mathbb{Z}_n$ on Q. Of course $\mathbb{Z}_n$ is also the centre of $SU(n)$. Thus if a generates $\mathbb{Z}_n$, it has a well defined action on $\hat{P}$ given by

$$(q, s) \to (qa, a^{-1}s)$$

where $q \in Q$ and $s \in SU(n)$. Set $P = \hat{P}/\mathbb{Z}_n$ and define an action of $\mathbb{Z}_4 \cdot U(n)$ on P as follows. Every element of $\mathbb{Z}_4 \cdot U(n)$ is either of the form g in $U(n)$ or γg in $\gamma \cdot U(n)$. Each g in $U(n)$ may be expressed as ut, where $u \in U(1)$ and $t \in SU(n)$. Of course, this expression is not unique, since $ut = (au)(a^{-1}t)$ for any $a \in \mathbb{Z}_n$. Then we set (recalling that $U(1) = \mathrm{Spin}(2)$)

$$R_g : \{q, s\} \to \{qu, st\}$$

where $\{q, s\}$ denotes the projection in P of (q, s) in $\hat{P}$. This is well defined, since $\{qua, sa^{-1}t\} = \{(qu)a, a^{-1}(st)\} = \{qu, st\}$. Next, we define

$$R_{\gamma g} : \{q, s\} \to \{q\gamma u, \overline{s}t\}$$

and this too is well defined since $\{qa\gamma u, (\overline{a})^{-1}\overline{s}t\} = \{q\gamma u\overline{a}, (\overline{a})^{-1}\overline{s}t\}$; we abuse the notation by using γ to denote the canonical element of order 4 in both $\mathbb{Z}_4 \cdot U(n)$ and $\mathrm{Pin}(2)$. As usual, we must check that the action of $\mathbb{Z}_4 \cdot U(n)$ is free and to the right. If $g = ut$ and if $\{q, s\}$ is a fixed point of g, then in $\hat{P}$ we must have $(qu, st) = (qa^m, a^{-m}s)$ for some integer m, and since Spin(2) acts freely on Q, this means $u = a^m$, $t = a^{-m}$ and so $g = 1$. Similarly, γg has no fixed point. To see that the action is to the right, we have, if $g_1 = u_1 t_1$ and $g_2 = u_2 t_2$

$$R_{\gamma g_1} R_{g_2}\{q, s\} = R_{\gamma g_1}\{qu_2, st_2\} = \{qu_2\gamma u_1, s\bar{t}_2 t_1\} = \{q\gamma \bar{u}_2 u_1, s\bar{t}_2 t_1\}$$

which indeed is the effect of $g_2\gamma g_1 = \gamma \bar{g}_2 g_1 = \gamma \bar{u}_2 u_1 \bar{t}_2 t_1$. Again

$$R_{\gamma g_1} R_{\gamma g_2}\{q, s\} = \{q\gamma u_2 \gamma u_1, s\bar{t}_2 t_1\} = \{q(-1)\bar{u}_2 u_1, s\bar{t}_2 t_1\}$$

consistent with $\gamma g_2 \gamma g_1 = -\bar{g}_2 g_1 = (-\bar{u}_2 u_1)\bar{t}_2 t_1$. Clearly

$$P/[\mathbb{Z}_4 \cdot U(n)] = Q/\mathrm{Pin}(2) = M$$

and one can show that the local triviality of Q implies that of P. Thus P is a principal $\mathbb{Z}_4 \cdot U(n)$ bundle over M. As we are assuming that Q is the holonomy bundle [13] of some connection, Q must be connected; hence the same is true of $\hat{P}$ and so of P. As Q has a connection which is not flat, $\dim(M) \geqslant 2$. The Hano–Ozeki–Nomizo theorem now completes the proof. □

From a physical point of view, this result is of interest because, in the standard model, the unbroken symmetry group is $U(3)$. The corresponding Pin-like group is of course $\mathbb{Z}_4 \cdot U(3)$. We now see that, in order to find examples of manifolds with principal bundles on which $\mathbb{Z}_4 \cdot U(3)$ occurs as a holonomy group, we need only to study the same problem for the much simpler group Pin(2).

An instructive and very simple example of a manifold on which Pin(2) occurs as a holonomy group is the real projective space $\mathbb{R}P^2$. The bundle of orthonormal frames over a two-dimensional Riemannian manifold M, denoted $O(M)$, is an $O(2)$ principal bundle over M. We say that M is a Pin manifold if there exists at least one Pin(2) bundle Pin(M) over M which is also a non-trivial $\mathbb{Z}_2$ bundle over $O(M)$, so that $\mathrm{Pin}(M)/\mathbb{Z}_2 = O(M)$ just as $\mathrm{Pin}(2)/\mathbb{Z}_2 = O(2)$. If M is orientable then Pin(M) always exists, but this is not necessarily so if M is not orientable. If M is neither orientable nor flat, then the holonomy group of the linear connection on $O(M)$ is precisely $O(2)$; thus, if M is a Pin manifold, the holonomy group of the pulled-back connection on Pin(M) must be Pin(2). This immediately gives us many examples: for example, the Klein bottle with metric slightly perturbed (so that it is not flat) is a Pin manifold, and so the above construction gives us a bundle with a connection having holonomy Pin(2).

Now in fact $\mathbb{R}P^2$ itself is *not* a Pin manifold [20]. Despite this, one can still construct other principal bundles over $\mathbb{R}P^2$ with connections having Pin(2) as a holonomy group. These facts can be seen in an elementary way as follows. Clearly $SO(3)$ acts transitively on $\mathbb{R}P^2$, with isotropy group $O(2)$, so $\mathbb{R}P^2 = SO(3)/O(2)$. In fact $SO(3)$ acts transitively on $O(\mathbb{R}P^2)$, with isotropy group $\mathbb{Z}_2$ (generated by the matrix diag$(1, -1, -1)$). Thus $O(\mathbb{R}P^2) = SO(3)/\mathbb{Z}_2$. Now if $O(\mathbb{R}P^2)$ lifted to a Pin(2) bundle over $\mathbb{R}P^2$, we would be able to exhibit Pin(2) as a subgroup of $SO(3)$, which is not possible, and this is a direct proof of the fact that $\mathbb{R}P^2$ is not Pin. While $SO(3)$ does not contain Pin(2) however, Spin(3) certainly does, and indeed Spin(3) is a Pin(2) bundle over $\mathbb{R}P^2$: we have $\mathrm{Spin}(3)/\mathrm{Pin}(2) = SO(3)/O(2) = \mathbb{R}P^2$. This is of course a version of the familiar Hopf bundle [21] over S^2. As Spin(3) is connected and $\mathbb{R}P^2$ is paracompact, we see that there is a connection on Spin(3) (as a bundle) with holonomy Pin(2); and so Pin(2) does occur as a holonomy group over $\mathbb{R}P^2$, despite the fact that $\mathbb{R}P^2$ is not a Pin manifold. In fact, there are infinitely many examples of this kind. For let r be any odd integer. Then $\mathrm{Pin}(2)/\mathbb{Z}_r$ is again isomorphic to Pin(2), and $\mathrm{Spin}(3)/\mathbb{Z}_r$ is a connected Pin(2) bundle over $\mathbb{R}P^2$; thus for each odd integer we obtain a distinct principal bundle, each one having a connection with holonomy Pin(2). (This does not work if r is even, for in that case $\mathrm{Pin}(2)/\mathbb{Z}_r = O(2)$; we have $O(\mathbb{R}P^2) = \mathrm{Spin}(3)/\mathbb{Z}_4$.) In all of these cases, theorem 6 gives us bundles over $\mathbb{R}P^2$ with connections having holonomy $\mathbb{Z}_4 \cdot U(m)$, for all odd m. Notice that $\mathbb{R}P^2$ has *no* quaternionic cover, and recall that its double cover, S^2, has *no* antipodal functions with respect to Spin(2).

We may summarize as follows. The problem of constructing a bundle with a connection having a holonomy group isomorphic to G is closely related to the analogous problem for G/N, where N is normal in G. We have found, for example, that solving this problem for Pin(2) allows us to solve it for the infinite family of groups $\mathbb{Z}_4 \cdot U(m)$ with m odd.

7. Cases where Pin(2) cannot be a holonomy group

Thus far, we have concentrated on using techniques which allow us to exhibit disconnected groups as holonomy groups of connections over manifolds satisfying the holonomy covering condition. In some cases, however, all such techniques must fail, since it is possible to prove that a specific disconnected group *cannot* occur as a holonomy group over some manifold, even though the latter does satisfy the holonomy covering condition. We have the following result.

Theorem 7. Let M be a connected paracompact manifold of dimension at least two, and suppose that M has a double cover. Assume further that the first and second homology groups (with integer coefficients) of every double cover are finite. Then there exists a principal bundle over M with a connection having a holonomy group isomorphic to Pin(2) if and only if M admits a quaternionic cover.

Remark. Note that the assumptions require the finiteness of the first two homology groups of *every* double cover. A manifold can have two distinct double covers, one having $H_1(\overline{M}, \mathbb{Z})$ and $H_2(\overline{M}, \mathbb{Z})$ finite and the other not. Let V be the compact orientable flat manifold [19] of the form $T^3/[\mathbb{Z}_2 \times \mathbb{Z}_2]$, which has $\overline{V} = T^3/\mathbb{Z}_2$ as double cover and consider $M = V \times \mathbb{R}P^3$. This manifold has $\overline{V} \times \mathbb{R}P^3$ and $V \times S^3$ as distinct double covers. Now $H_1(V, \mathbb{Z})$ is finite, and so are $H_1(V \times S^3, \mathbb{Z})$ and (by Poincaré duality and the Künneth formula) $H^2(V \times S^3, \mathbb{Z})$, the second cohomology group; thus $H_2(V \times S^3, \mathbb{Z})$ is also finite. But $H_1(\overline{V} \times \mathbb{R}P^3, \mathbb{Z})$ is infinite, since [19] $H_1(\overline{V}, \mathbb{Z})$ is infinite.

Proof. The sufficiency of the existence of a quaternionic cover is given by corollary 5.2. Suppose on the other hand that P is a bundle over M with a connection having holonomy Pin(2). By the holonomy reduction theorem [13], we may take it that P is a Pin(2) bundle, and that P is a holonomy bundle of the given connection; hence P must be connected, and the same is true of $\overline{M} = P/\mathrm{Spin}(2)$. Thus $\overline{M}$ is a non-trivial double cover of M, and so, by assumption, $H_1(\overline{M}, \mathbb{Z})$ and $H_2(\overline{M}, \mathbb{Z})$ are finite. Hence, in particular, they are finitely generated, and so, by the universal coefficient theorem [21] $H^2(\overline{M}, \mathbb{Z})$ is just $F_2 \oplus T_1$, where F_2 is the free part of $H_2(\overline{M}, \mathbb{Z})$ and T_1 is the torsion of $H_1(\overline{M}, \mathbb{Z})$. Thus $H^2(\overline{M}, \mathbb{Z})$ is finite and isomorphic to $H_1(\overline{M}, \mathbb{Z})$. Now principal $U(1)$ bundles over $\overline{M}$ are classified by $H^2(\overline{M}, \mathbb{Z})$; the element of $H^2(\overline{M}, \mathbb{Z})$ corresponding to a particular $U(1)$ bundle is just the first Chern class [22] of that bundle. In our case, the $U(1)$ bundles over $\overline{M}$ are constructed from the finite group $H_1(\overline{M}, \mathbb{Z})$ as follows. Let W be the fundamental group of $\overline{M}$, and let W' be its commutator subgroup. Set $M' = \widetilde{M}/W'$, where $\widetilde{M}$ is the universal cover of M, so that $\overline{M} = \widetilde{M}/W = M'/H_1(\overline{M}, \mathbb{Z})$. Now $H_1(\overline{M}, \mathbb{Z})$ is a finite Abelian group, a product of finite cyclic groups. Factoring M' by all but one of these cyclic groups, we obtain a 'cyclic cover' M^c, that is, a manifold such that $M^c/\mathbb{Z}_r = \overline{M}$. For example, if $\pi_1(\overline{M}) = \mathbb{Z}_2 \times \mathbb{Z}_2$, then $H_1(\overline{M}, \mathbb{Z}) = \mathbb{Z}_2 \times \mathbb{Z}_2$, and it is clear that $\overline{M}$ has three $\mathbb{Z}_2$ covers. Given a specific cyclic cover M^c, we let $\mathbb{Z}_r$ act on $M^c \times U(1)$ by $(m, u) \to (a(m), au)$, where a generates $\mathbb{Z}_r$ and then $[M^c \times U(1)]/\mathbb{Z}_r$ is a non-trivial $U(1)$ bundle over $\overline{M}$. All of the non-trivial $U(1)$ bundles over $\overline{M}$ are constructed in this way. Thus, each element of our bundle P has the form $\{m, u\}$, where $m \in M^c$, a certain cyclic cover of $\overline{M}$, $u \in U(1)$, and the brackets indicate the factoring by a particular cyclic group $\mathbb{Z}_r$ such that $M^c/\mathbb{Z}_r = \overline{M}$. Let μ be the involution on $\overline{M}$ such that $\overline{M}/\mathbb{Z}_2 = M$; then μ is covered by some fixed-point-free map $b : M^c \to M^c$. Setting $\mathrm{Pin}(2) = \mathrm{Spin}(2) \cup \gamma \cdot \mathrm{Spin}(2)$, we see that the right action of γ on P must take the form $R_\gamma\{m, u\} = \{b(m), u^*\}$ for some u^*. Let y, the projection of m to $\overline{M}$, be fixed. Letting u vary, we obtain a map f_y which is defined by $f_y(u) = u^*$; as the notation suggests, f_y may depend on y. Now for any $v \in U(1)$, we have $v\gamma = \gamma\overline{v}$, and so $\{b(m), (uv)^*\} = \{b(m), u^*\overline{v}\}$, hence $f_y(uv) = f_y(u)\overline{v}$ for all u, v. But by definition of the quotient,

$$\{b(m),\, f_y(u)\} = \{ba(m),\, f_y(au)\} = \{ba(m), \overline{a}\, f_y(u)\} = \{aba(m),\, f_y(u)\}$$

hence $aba = b$ and so $bab^{-1} = a^{-1}$. Evidently b^2 commutes with a and covers the identity map on $\overline{M}$, and thus $b^2 = a^p$ for some integer p. Thus, we have shown that M^c admits a fixed-point-free action by a group satisfying $bab^{-1} = a^{-1}$ and $b^2 = a^p$, such that the quotient is M; that is, we have shown that M has a quaternionic cover, provided that $p \neq 0$. To rule out this last possibility, we note that, with the above notation, $f_y(uv) = f_y(u)\overline{v}$ implies $f_y(v) = f_y(1)\overline{v}$, or $f_y(v) = f(y)\overline{v}$ if we define $f_y(1) = f(y)$. Then

$$R_{\gamma^2}\{m, u\} = R_\gamma\{b(m),\, f(y)\overline{u}\} = \{b^2(m),\, f(\mu(y))\overline{f(y)}u\} = \{m, -u\}.$$

Hence if $p = 0$, so that b^2 is the identity map, we see that f must be an antipodal function on $\overline{M}$. Now recalling that $\overline{M}$ is connected, let y be any point in $\overline{M}$ and let c be a curve from y to $\mu(y)$, so that $\mu \circ c$ is a curve from $\mu(y)$ to y. If we use f to attach an element of $U(1)$ to each point on c, then clearly the relation $f(\mu(y)) = -f(y)$ implies that any element of $U(1)$ which fails to occur along c must occur along $\mu \circ c$. Thus f defines a map from $\overline{M}$ to the circle S^1, and this map is not homotopic to the trivial map. Now homotopy classes of maps from $\overline{M}$ to S^1 are classified [23] by the cohomology group $H^1(\overline{M}, \mathbb{Z})$. But since we are assuming that $H_1(\overline{M}, \mathbb{Z})$ the corresponding homology group is finite, $H^1(\overline{M}, \mathbb{Z}) = 0$ and so every map $\overline{M} \to S^1$ is homotopic to the trivial map. This contradiction completes the proof. □

If M is orientable and three-dimensional, then Poincaré duality gives us the finiteness of $H^2(\overline{M}, \mathbb{Z})$ directly from that of $H_1(\overline{M}, \mathbb{Z})$, and so we have the following result.

Corollary 7.1. Let M be any orientable paracompact 3-fold with $H_1(\overline{M}, \mathbb{Z})$ finite for each double cover $\overline{M}$. Then Pin(2) occurs as a holonomy group over M if and only if M has a quaternionic cover.

For example, the spherical space form $S^3/\widetilde{O}_{48}$, where $\widetilde{O}_{48}$ is the binary octahedral group [19], satisfies the holonomy covering condition for Pin(2) (its double cover is $S^3/\widetilde{T}_{24}$, where $\widetilde{T}_{24}$ is the binary tetrahedral group); but as it has no quaternionic cover, and since $H_1(S^3/\widetilde{T}_{24}, \mathbb{Z}) = \mathbb{Z}_3$, Pin(2) cannot occur as a holonomy group in this case. This is our first example in which the holonomy covering condition definitely fails to be sufficient.

Another interesting source of examples is provided by the connected compact simply connected Lie groups G. Here the second homotopy group $\pi_2(G)$ always vanishes, and so

$H_2(G, \mathbb{Z}) = 0$ by the Hurewicz isomorphism theorem [21]. Thus if Γ is any finite subgroup of G, then $H_1(G/\Gamma, \mathbb{Z})$ and $H_2(G/\Gamma, \mathbb{Z})$ are finite, and so we have the following result.

Corollary 7.2. Let G be a connected, compact, simply connected Lie group, and let Γ be a finite subgroup of G. Then Pin(2) occurs as a holonomy group over G/Γ if and only if Γ has a normal subgroup N such that $\Gamma/N = Q_{4n}$ for some $n \geqslant 1$.

For example, each group $SO(n)$, $n \geqslant 3$, satisfies the holonomy covering condition for Pin(2), but Pin(2) does not occur as a holonomy group over any such manifold. The same is true of the groups $SO(4n)/\mathbb{Z}_2 = \text{Spin}(4n)/[\mathbb{Z}_2 \times \mathbb{Z}_2]$, but Pin(2) does occur over $SO(4n+2)/\mathbb{Z}_2 = \text{Spin}(4n+2)/\mathbb{Z}_4$, since Spin$(4n+2)$ is the $Q_4 = \mathbb{Z}_4$ cover of this group.

Finally, the simplest manifolds satisfying the holonomy covering condition for all of the Pin groups are the real projective spaces $\mathbb{R}P^m$. When $n \geqslant 3$, Pin(n) is treated in corollary 4.1. We saw in section 6 that Pin(2) occurs as a holonomy group over $\mathbb{R}P^2$ (which is not affected by theorem 7 since $H_2(S^2, \mathbb{Z})$ is not finite). For $m \geqslant 3$, $H_1(S^m, \mathbb{Z})$ and $H_2(S^m, \mathbb{Z})$ are both trivial, and so theorem 7 covers the remaining cases, and we obtain the following statement.

Corollary 7.3. There exists a principal bundle over $\mathbb{R}P^m$, $m \geqslant 2$, with a connection having holonomy group isomorphic to Pin(n), $n \geqslant 2$, if and only if either $n \geqslant 3$, *or* $n = m = 2$.

Thus, for example, if the topology of spacetime is $\mathbb{R}\times\mathbb{R}P^3$, then 'Alice electrodynamics' [7] based on the Pin(2) subgroup of Spin(10) is *not* possible, despite the fact that the holonomy covering condition is satisfied—an unexpected result. Note that theorem 6 yields a similar conclusion for all of the groups $\mathbb{Z}_4 \cdot U(n)$ with n odd.

8. Conclusion

Connections with disconnected holonomy groups correspond to an interesting type of non-perturbative gauge configuration. As with monopoles, instantons and so on, the topology of the underlying spacetime (or generalized spacetime) plays a central role in determining whether disconnected gauge holonomy groups are possible in any given case. For some groups (such as Pin$(n), n \geqslant 3$) the obvious necessary condition also suffices, but for others (such as Pin(2) and $\mathbb{Z}_4 \cdot U(n)$, n odd) it does not.

In this work we have presented a variety of techniques for constructing gauge configurations with disconnected holonomy groups. These allow us to deal with all disconnected groups G of type II, and many groups of type III. Using these methods, one can give a complete analysis of the existence of such gauge configurations over all homogeneous, locally isotropic Riemannian manifolds, for physically interesting subgroups of grand unification groups such as Spin(10); that is, we can give an existence theory for 'Alice cosmology'. This will be presented elsewhere.

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